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A foray into the demarcation of the Gini coefficient[☆]

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ABSTRACT

We specify the domain in the income distribution that includes the people to whom income transfers will not increase inequality in that income distribution. Inspired by Sen's (1973, 1997) characterization of the Gini coefficient as a ratio between a measure of aggregate income-based "depression" (stress) and aggregate income, we inquire as to whether in the wake of an increase of an income or of incomes in a given income distribution, the Gini coefficient does not increase. To this end, we identify the corresponding "safe" domain and show that the pivotal value that demarcates this domain can be elicited from a simple linear function of the Gini coefficient itself. Our rule of demarcation provides for policy interventions that seek to increase a particular income or particular incomes while not exacerbating inequality in the income distribution as measured by the Gini coefficient.

1. Introduction

Governments in developed and developing countries alike exhibit considerable sensitivity to inequality in income distributions and institute income transfers to low-income groups. These two characteristics are not unrelated: governments that enact the latter seek not to exacerbate, and possibly seek to reduce, the former. For example, in the US "inequality is [considered] an urgent problem" and is addressed by transfers to "poor and low-income families" (Peterson Institute for International Economics, 2020). A main objective of the transfers in Brazil's Bolsa Familia (which in terms of the number of beneficiaries is perhaps the largest cash transfer program in the developing world) is the "reduction of poverty and inequality" (Sanchez-Ancochea and Mattei, 2011). Although in this paper we refer to income as the variable of interest, income does not need to be the only such variable. For example, Lipton (1985) and Faguet et al. (2020) have advocated increasing the farmland of poor smallholders who are defined by the size of their farms. This measure of poverty is distinct from the degree of inequality in the distribution of farms by size (the inequality in landholdings as measured by what Acemoglu et al. (2008) term "the Land Gini"). Governments that control arable land can transfer public land to tenants and nearly-landless farmers as Colombia, for example, has done for two centuries. As has been emphasized, governments' land transfer policies should "attack poverty" in villages in the developing world without exacerbating intravillage landholding inequality (inequality in the distribution of lands). An obvious question is: who precisely are the poor and low-income people to whom transfers will not increase inequality in the income distribution?

In this paper, we formulate a criterion that identifies the domain in the income distribution that includes people to whom income transfers will not increase inequality in the income distribution, and we show that this criterion is based entirely on the measure of income inequality itself. Our findings have several repercussions. First, the split of domains in the income distribution does not occur, as might intuitively be expected, at the

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center of the income distribution (it is located at the center or above the center of the income distribution, cf. Remark 3 below). Second, the higher the degree of income inequality as measured by the Gini coefficient, the larger the domain of people to whom income transfers will not increase the coefficient (cf. Remark 4 below). It is worth clarifying that our analysis differs from treatments of income transfers within the existing income distribution. That is, we do not concern ourselves with the redistribution of incomes. What we study are transfers from the outside, so to speak, as in the examples of Brazil's Bolsa Familia and the transfer of public land to poor farmers.

Our point of departure is Sen's (1973) representation of the Gini coefficient as a ratio of two measures of income: relative and absolute. Introducing the Gini coefficient, Sen (1973, p. 33) writes: "In any pair-wise comparison the man with the lower income can be thought to be suffering from some depression on finding his income to be lower. Let this depression be proportional to the difference in income. The sum total of all such depressions in all possible pair-wise comparisons takes us to the Gini coefficient." Specifically, we reformulate Sen's representation of the Gini coefficient, expressing the coefficient as a ratio of aggregate stress ("depression" in Sen's representation) and aggregate income.[1] Drawing on this reformulation, for a given income distribution we identify, first, the domain of the distribution in which, upon a rank-preserving increase of any income, the Gini coefficient does not increase and, second, by implication, the complementary domain in which it increases. We show that the pivotal value that demarcates the "safe" domain can be elicited from a simple linear function of the Gini coefficient itself: the Gini coefficient is the exclusive basis for the demarcation. This is intriguing because the Gini coefficient is found to "act" doubly: it is a measure of inequality of an income distribution, and it is a marker of domains of the income distribution. We also generalize our result and prove that increasing multiple incomes, not necessarily in a rank-preserving manner, does not increase the Gini coefficient as long as all the recipients of the increases of incomes stay below the pivotal value that demarcates the "safe" domain.

Our finding identifies for governments considering increasing a particular income or incomes, while at the same time not exacerbating inequality in the income distribution as measured by the Gini coefficient, the "safe" zone for such a policy.

2. Introductory example: The two-person case

Consider population N of individuals 1 and 2, $N=\{1,2\}$, with income vector $y=(y_1,y_2)$ such that $0<y_1<y_2$. The Gini coefficient, $G(y_1,y_2)$, for this population is

$$G(y_1,y_2)=\frac{\frac{1}{2}(y_2-y_1)}{y_1+y_2}. \tag{1}$$

The term in the denominator is the population's aggregate or total income, which throughout this paper we denote by TI. The term in the numerator is a measure of the income-based stress, denoted throughout by AS. In (1), the stress experienced by individual 1 is the income gap to which individual 1 is subjected, normalized by the size of the population. Thus, in (1), the Gini coefficient, which we denote by G, is expressed as $G=\frac{AS}{TI}$.

Remark 1. We show that (1) is a particular case of Sen's (1973) representation of the Gini coefficient. In population $N=\{1,2,\ldots,n\}$, $n\geq 2$, let $y=(y_1,\ldots,y_n)$ be the vector of incomes of the members of the population, and let these incomes be ordered $0<y_1<y_2<\ldots<y_n$. As per Sen (1973), the Gini coefficient can be expressed as

$$G\equiv\frac{\sum_{j=1}^{n}\sum_{i=1}^{n}\left|y_i-y_j\right|}{2n^2\bar{y}}, \tag{2}$$

where $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$ is the mean income of population N.

Noting that $\sum_{j=1}^{n}\sum_{i=1}^{n}\left|y_i-y_j\right|=2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j-y_i)$, an equivalent representation of G in (2), which disposes of the need to operate with absolute values, is

$$G=\frac{\frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j-y_i)}{\sum_{i=1}^{n}y_i}. \tag{3}$$

When $n=2$, (3) reduces to (1) because then $\frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j-y_i)=\frac{1}{2}(y_2-y_1)=AS$, and $\sum_{i=1}^{n}y_i=y_1+y_2=TI$. E.O.R.[2]

In the representation of the Gini coefficient in (1), a rank-preserving increase of the lower income y_1 reduces AS and increases TI. As a result, the Gini coefficient decreases. This is seen in (1) straightforwardly: the numerator and the denominator change in opposite directions.

To determine the effect on G of an increase of the higher income y_2, the Gini coefficient in (1) can be rewritten as

$$G(y_1,y_2)=\frac{1}{2}\cdot\frac{1-\frac{y_1}{y_2}}{1+\frac{y_1}{y_2}}. \tag{1'}$$

[1] As emerges from the insightful account by Ceriani and Verme (2012), despite being not only a statistician but also a sociologist and a demographer, Gini (1912) developed a mathematical formula for measuring dispersion independently of social-psychological principles and preferences. That formula turned out to be a widely used measure of inequality. Sen (1973, p. 149) remarks: "[T]he Gini coefficient [is] still the most commonly used measure of inequality in empirical work." Establishing the link of the Gini coefficient with income-based depression and stress is due to Sen (1973, 1997).

[2] In order not to let the text of a remark mingle with the subsequent main text, at the end of each remark, including this remark, we write E.O.R.

When y_2 increases, the numerator in the second fraction of (1') becomes bigger, and the denominator in the second fraction of (1') becomes smaller. As a result, G increases. In this two-person case, the "safe" domain of the distribution in which, upon a rank-preserving increase of income, the Gini coefficient does not increase consists of income y_1.

3. A criterion for the Gini coefficient not to increase upon a rank-preserving increase of the income of an individual

Consider a population $N=\{1,2,\ldots,n\}$ with an income vector $y=(y_1,y_2,\ldots,y_n)$ such that $0<y_1<y_2<\ldots<y_n$. The aggregate stress of this population of n individuals is

$$AS(y)=\frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j-y_i),$$

and the aggregate income of this population of n individuals is

$$TI(y)=\sum_{i=1}^{n}y_i.$$

Consequently, the Gini coefficient is the ratio

$$G(y)=\frac{AS(y)}{TI(y)}.$$

For a given income distribution, we seek to determine the domain where a rank-preserving increase of an income does not result in an increase of the Gini coefficient.

Claim 1. The Gini coefficient does not increase in the wake of a rank-preserving increase of income y_k, that is, the income of individual k, if and only if the position of this individual in the income distribution is such that

$$k\leq\frac{n(G(y)+1)+1}{2}. \tag{4}$$

Proof. In the Appendix.

Corollary. The Gini coefficient increases in the wake of a rank-preserving increase of the income y_k of individual k if and only if the position of this individual in the income distribution is such that

$$k>\frac{n(G(y)+1)+1}{2}.$$

Remark 2. Given n, the right-hand side of (4) depends only on the value of the Gini coefficient of the initial income vector, $G(y)$. Then Claim 1 reveals that the domain of the individuals for whom a rank-preserving increase of income does not increase the Gini coefficient is determined by the Gini coefficient of the initial income vector itself. E.O.R.

Remark 3. What (4) also implies is that a pivotal value (a value that divides the income distribution into two domains), k_0, exists and that this value is equal to $k_0=\left\lfloor\frac{n(G+1)+1}{2}\right\rfloor$, where by $\lfloor x\rfloor$ we denote the integer part of x, such that the Gini coefficient will not increase upon a rank-preserving increase of the income of individual k if and only if $k\leq k_0$. We now identify the values that k_0 can assume for any given n. To this end, we note, bearing in mind our assumption $0<y_1<y_2<\ldots<y_n$ (which means that it cannot hold that all incomes are the same, so the Gini coefficient is nonzero, and that it also cannot hold that all incomes but one are zero, so the Gini coefficient is smaller than $\frac{n-1}{n}$) that the Gini coefficient can take any value in the interval $\left(0,\frac{n-1}{n}\right)$.[3] Thus,

$$\frac{n+1}{2}=\frac{n(0+1)+1}{2}<\frac{n(G(y)+1)+1}{2}<\frac{n\left(\frac{n-1}{n}+1\right)+1}{2}=n.$$

From the preceding line we see that $\frac{n(G(y)+1)+1}{2}$ can take any of the values that reside in the interval $\left(\frac{n+1}{2},n\right)$. Then there exist numbers $a>0$ and $b>0$ such that

$$\frac{n+1}{2}+a=\frac{n(G(y)+1)+1}{2}=n-b.$$

[3] In Stark (2024), we show that the $\frac{n-1}{n}$ upper bound on G is not inevitable. We draw on the presentation by Sen (1973, 1997) of the Gini coefficient of income inequality in a population that we already cited in the introduction of the current paper: "In any pair-wise comparison … ." Sen's verbal account is accompanied by a formula Sen (1997, p. 31, eq. 2.8.1). The formula yields a coefficient bounded from above by a number smaller than 1. This creates a difficulty, because the "mission" of a measure of inequality defined on the unit interval is to assign 0 to perfect equality (maximal equality) and 1 to perfect inequality (maximal inequality). In Stark (2024), we show that when the Gini coefficient is elicited from a neat measure of the aggregate income-related depression of the population that consists of people who experience income-related depression, the obtained Gini coefficient (that is, our alternative version of the Gini coefficient) is "well behaved" in the sense that it is bounded from above by 1.

We note[4] that $k_0=\left\lfloor\frac{n(G(y)+1)+1}{2}\right\rfloor=\left\lfloor\frac{n+1}{2}+a\right\rfloor\geq\left\lfloor\frac{n+1}{2}\right\rfloor\geq\frac{n}{2}$ and that it is possible that $k_0=\left\lfloor\frac{n(G(y)+1)+1}{2}\right\rfloor=\left\lfloor\frac{n+1}{2}+a\right\rfloor=\frac{n}{2}$ (when n is even and $a<\frac{1}{2}$). Thus, $\frac{n}{2}$ is the minimum of the set of possible values of k_0. Concurrently, $k_0=\left\lfloor\frac{n(G(y)+1)+1}{2}\right\rfloor=\lfloor n-b\rfloor\leq n-1$, that is, $k_0\leq n-1$.
In sum: k_0 can assume any integer value that falls in the interval $\left[\frac{n}{2},n-1\right]$, meaning that the pivotal value that demarcates the income distribution into two domains is located at the center of the income distribution or above the center of the income distribution. E.O.R.

Remark 4. There is another insight that $k_0=\left\lfloor\frac{n(G+1)+1}{2}\right\rfloor$ gives rise to. Because k_0 is weakly increasing in G, it becomes easier for the constraint $k\leq k_0$ to hold when G is higher. This means that as G rises, the domain for which a rank-preserving increase of an income results in the coefficient not increasing eventually becomes bigger.

Example 1. The income distribution is such that the income of individual i is i.

To demonstrate an application of Claim 1, we present as an example the case of a population of n individuals where the income of individual i is i, $i=1,2,\ldots,n$ (the income of an individual is then the rank of the individual, which, in turn, can be referred to as the name of the individual). In this case, $TI(1,2,\ldots,n)$ and $AS(1,2,\ldots,n)$ take the following values[5]:

$$TI(1,2,\ldots,n)=\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$$

and

$$AS(1,2,\ldots,n)=\frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(j-i)=\frac{1}{n}\left\{\sum_{j=2}^{n}(j-1)+\sum_{j=3}^{n}(j-2)+\ldots+\sum_{j=n-1}^{n}[j-(n-2)]+[n-(n-1)]\right\}$$
$$=\frac{1}{n}\left(\sum_{j=1}^{n-1}j+\sum_{j=1}^{n-2}j+\ldots+\sum_{j=1}^{2}j+\sum_{j=1}^{1}j\right)=\frac{1}{n}\sum_{i=1}^{n-1}\frac{i(i+1)}{2}=\frac{1}{2n}\sum_{i=1}^{n-1}(i+i^2)=\frac{1}{2n}\left(\frac{(n-1)n}{2}+\frac{(n-1)n(2n-1)}{6}\right)=\frac{(n-1)(n+1)}{6}.$$

Then $G(1,2,\ldots,n)=\frac{\frac{(n-1)(n+1)}{6}}{\frac{n(n+1)}{2}}=\frac{n-1}{3n}$. Consider individual $k\in\{1,2,\ldots,n\}$. When there is a rank-preserving increase of the income of this individual, the condition for the Gini coefficient to not increase, that is, condition $k\leq\frac{n(G(1,2,\ldots,n)+1)+1}{2}$ as per Claim 1, takes the form

$$k\leq\frac{n\left(\frac{(n-1)}{3n}+1\right)+1}{2}=\frac{2n+1}{3}.$$

If $k>\frac{2n+1}{3}$, then a rank-preserving increase of the income of individual k will increase the Gini coefficient. In this particular case where the income of individual i is i, then for a large n, a rank-preserving increase of the income of an individual in (approximately) the bottom two-thirds of the income distribution will not increase the Gini coefficient, whereas a rank-preserving increase of the income of an individual in (approximately) the top third of the income distribution will increase the Gini coefficient. To illustrate this, let $n=100$. Then $\frac{2n+1}{3}=67$. Therefore, for $k\leq 67$, a rank-preserving increase of the income of individual k does not result in an increase of the Gini coefficient; and for $k>67$, a rank-preserving increase of the income of individual k results in an increase of the Gini coefficient.

Remark 5. In routine applications of the Gini coefficient, the income distributions are of large populations. When n is large, it is more convenient to use the pivotal ratio $q_0=\frac{k_0}{n}$ of the individuals in the population for whom a rank-preserving increase of income does not result in an increase of the Gini coefficient than to use the pivotal value k_0. Because

$$k_0=\left\lfloor\frac{n(G(y)+1)+1}{2}\right\rfloor\approx\frac{n(G(y)+1)+1}{2},$$

then

$$q_0=\frac{k_0}{n}\approx\frac{G(y)+1+\frac{1}{n}}{2}\approx\frac{G(y)+1}{2},$$

where for a sufficiently large n we obviously have that $\frac{1}{n}\approx 0$. Therefore, for any large population for which the Gini coefficient is known, we can estimate by $\frac{G(y)+1}{2}$ the pivotal ratio of the individuals in the population for whom a rank-preserving increase of an income will result in the Gini coefficient not increasing. E.O.R.

[4] The inequality $\left\lfloor\frac{n+1}{2}\right\rfloor\geq\frac{n}{2}$ holds because either n is even, and then $\left\lfloor\frac{n+1}{2}\right\rfloor=\frac{n}{2}$, or n is odd, and then $\frac{n+1}{2}$ is already an integer, so $\left\lfloor\frac{n+1}{2}\right\rfloor=\frac{n+1}{2}>\frac{n}{2}$.

[5] In the calculations that follow, we draw on the summation formulas $\sum_{i=1}^{l}i=\frac{l(l+1)}{2}$ and $\sum_{i=1}^{l}i^2=\frac{l(l+1)(2l+1)}{6}$ for any $l\in\mathbb{N}$.

Example 2. A real-world example.

According to EUROSTAT, in 2023, the Gini coefficient for Poland was 0.27.[6] Thus, the pivotal ratio for Poland is

$$\frac{0.27+1}{2}=0.635,$$

meaning that a rank-preserving increase of the income of any individual whose income places him among the lowest 63.5 percent of the income earners of the population of Poland will not result in an increase of the Gini coefficient of Poland.

Remark 6. It might be argued that our analysis is biased because the "business" of a government is to improve social welfare rather than avoid exacerbating inequality in the income distribution. But in our case this is not really an issue. A simple way of demonstrating this is to "enlist" the social welfare function proposed by Sen (1973, 1997), Sen (1976, 1982), denoted here by SWF_{SEN}. Sen defined $SWF_{SEN}\equiv\mu(1-G)$, where μ is income per capita and G is the Gini coefficient. In our setting, as per $k\leq\frac{n(G(y)+1)+1}{2}$ in Claim 1, G decreases or remains as is, and μ increases, so social welfare improves. E.O.R.

4. An extension: A criterion for the Gini coefficient not to increase upon concurrent rank-preserving increases of the incomes of several individuals

In the setting analyzed in the preceding sections, the rank-preserving increase is of a single income taken from the range of incomes as per Claim 1. However, our analysis also applies to settings in which the increase is of the incomes of several individuals from that range and in which the increase does not have to be rank-preserving. Thus, the reach of Claim 1 can be extended: increases of the incomes of $l\in\{1,2,\ldots,n\}$ individuals $k_1,k_2,\ldots,k_l$, that is, concurrent multiple increases, entail a decrease or no change in the Gini coefficient as long as the position of each of these individuals in the income distribution, both before and after the increase, abides by $k_i\leq\frac{n(G(y)+1)+1}{2}$ for $i\in\{1,2,\ldots,l\}$. In the following claim, we formalize this insight.

Claim 2. Let $l\in\{1,2,\ldots,n\}$, and let $k_i\leq\frac{n(G(y)+1)+1}{2}$ for $i\in\{1,2,\ldots,l\}$. Then, in the wake of increases of the incomes of individuals $k_1,k_2,\ldots,k_l$, the Gini coefficient does not increase as long as the ranks $\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_l$ of individuals $k_1,k_2,\ldots,k_l$ in the distribution of incomes arising after the increase also satisfy $\tilde{k}_i\leq\frac{n(G(y)+1)+1}{2}$ for $i\in\{1,2,\ldots,l\}$.

Proof. In the Appendix.

Remark 7. As the case of $l=1$ is not excluded, Claim 2 and its proof reveal that the assumption that the increase of incomes is rank-preserving is also not necessary for Claim 1 to hold as long as inequality (4) is satisfied by the rank of individual k both before and after the increase. E.O.R.

Remark 8. By means of an example, we explain why we can sweep aside the assumption that the increase of incomes has to be rank-preserving. We consider a population endowed with a vector of incomes $y=(1,3,6,10)$. The Gini coefficient of this population is $G(y)=0.375$. We increase the income of individual 1 by 4, which results in a new vector of incomes $\tilde{y}=(5,3,6,10)$. While this increase satisfies the assumptions of Claim 2, it is not rank-preserving. However, we can pair up this increase with the following rank-preserving increase of incomes: an increase of the income of individual 1 by 2, and an increase of the income of individual 2 by 2. Proceeding in this manner, we obtain another vector of incomes $\tilde{y}_\Phi=(3,5,6,10)$, which is just a permutation (a change of ordering) of vector $\tilde{y}$. In particular, the increase of incomes to $\tilde{y}_\Phi$ also satisfies the assumptions of Claim 2. Moreover, $G(\tilde{y})=G(\tilde{y}_\Phi)$, and, thus, every claim (such as Claim 2) about the value of the Gini coefficient of $\tilde{y}_\Phi$ carries over to the value of the Gini coefficient of $\tilde{y}$. In the Appendix, we formally prove that any increase of incomes satisfying the assumptions of Claim 2 can be paired up in this same manner with a rank-preserving increase of incomes that also satisfies the assumptions of Claim 2, and that leads to the same Gini coefficient. E.O.R.

The generalization from Claim 1 to Claim 2 bears importantly on real-world applications. A program that aims to reduce inequality in the distribution of incomes by increasing the incomes of relatively poor individuals might be costly or difficult to implement if it were necessary to ensure that the increases are rank-preserving. It is easier to see to it that no individual whose income is increased will move to a position above a pivotal value in the hierarchy of incomes. Recalling Example 2, any increase of incomes of the bottom 63.5% income earners in Poland in 2023 will not increase the Gini coefficient as long as no recipient of an income transfer advances to the domain of the top 36.5% of income earners.

5. Conclusion

Sen's (1973 and 1997) conceptualization of the Gini coefficient as a ratio between aggregate social-psychological income-based depression or stress and total income gives rise to several novel "Gini-based" insights. In Stark (2025), we transform the Gini coefficient to a social welfare function. This conversion is revealing because it goes further than incorporating the Gini coefficient as an input in a social welfare function. Moreover, the "Gini social welfare function" has a desirable property not possessed by a social welfare function in which the Gini coefficient features as an input. (This property is the capability to assign weights in the function to aggregate income and income-based stress and adapt these weights to a population's preferences, resulting in a measure of social welfare that duly reflects the manner in which people assess their wellbeing.) In this paper, we transform the Gini coefficient into a marker that demarcates a given income distribution into two mutually exclusive domains such that in one domain an increase of an income decreases the coefficient or leaves it as is, whereas in the other domain it increases the coefficient. It is intriguing that after more than a century of using and commenting on the Gini coefficient, it is still possible to identify new features and new roles of the coefficient.

[6] https://ec.europa.eu/eurostat/databrowser/view/tessi190/default/table?lang=en

Appendix: Proofs of Claims 1 and 2

Proof of Claim 1. For $k\in\{1,2,\dots,n\}$ and $y=(y_1,y_2,\dots,y_n)$ we can express $AS(y)$ and $TI(y)$, respectively, as follows:

$$AS(y)=\frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j-y_i)=\frac{1}{n}\left[g_1(y_1,y_2,\dots,y_{k-1},y_{k+1},\dots,y_n)+\sum_{j=k+1}^{n}(y_j-y_k)+\sum_{j=1}^{k-1}(y_k-y_j)\right],$$

where $g_1(y_1,y_2,\dots,y_{k-1},y_{k+1},\dots,y_n)$ is the component of $AS(y)$ that does not depend on y_k, and

$$TI(y)=\sum_{i=1}^{n}y_i=g_2(y_1,y_2,\dots,y_{k-1},y_{k+1},\dots,y_n)+y_k,$$

where $g_2(y_1,y_2,\dots,y_{k-1},y_{k+1},\dots,y_n)$ is the component of $TI(y)$ that does not depend on y_k.
Let the income of individual k increase by $\alpha>0$ in a rank-preserving manner: $y_k+\alpha<y_{k+1}$. Then

$$AS(y_1,\dots,y_{k-1},y_k+\alpha,y_{k+1},\dots y_n)=\frac{1}{n}\left[g_1(y_1,y_2,\dots,y_{k-1},y_{k+1},\dots,y_n)+\sum_{j=k+1}^{n}(y_j-y_k-\alpha)+\sum_{j=1}^{k-1}(y_k+\alpha-y_j)\right]$$
$$=AS(y)+\frac{1}{n}[(n-k)(-\alpha)+(k-1)\alpha]=AS(y)+\frac{2k-(n+1)}{n}\alpha.$$

At the same time, also

$$TI(y_1,\dots,y_{k-1},y_k+\alpha,y_{k+1},\dots y_n)=TI(y)+\alpha.$$

Thus,

$$G(y_1,\dots,y_{k-1},y_k+\alpha,y_{k+1},\dots y_n)-G(y)=\frac{AS(y_1,\dots,y_{k-1},y_k+\alpha,y_{k+1},\dots y_n)}{TI(y_1,\dots,y_{k-1},y_k+\alpha,y_{k+1},\dots y_n)}-\frac{AS(y)}{TI(y)}=\frac{AS(y)+\frac{2k-(n+1)}{n}\alpha}{TI(y)+\alpha}-\frac{AS(y)}{TI(y)}$$
$$=\frac{AS(y)TI(y)+\frac{2k-(n+1)}{n}\alpha TI(y)-AS(y)TI(y)-\alpha AS(y)}{(TI(y)+\alpha)TI(y)}=\frac{\frac{2k-(n+1)}{n}-\frac{AS(y)}{TI(y)}}{\frac{(TI(y)+\alpha)}{\alpha}}=\frac{\frac{2k-(n+1)}{n}-G(y)}{\frac{(TI(y)+\alpha)}{\alpha}}.$$

Because it always holds that $\frac{(TI(y)+\alpha)}{\alpha}>0$, then the condition $G(y_1,\dots,y_{k-1},y_k+\alpha,y_{k+1},\dots y_n)-G(y)\le 0$ (that is, the condition that the Gini coefficient does not increase) is equivalent to

$$\frac{2k-(n+1)}{n}-G(y)\le 0,$$

which in turn is equivalent to

$$k\le\frac{nG(y)+(n+1)}{2}=\frac{n(G(y)+1)+1}{2}.$$

Q.E.D.

Proof of Claim 2. To begin with, we note that it is sufficient to prove Claim 2 for the case of rank-preserving increases of incomes. To this end, we denote the income vector of population $N=\{1,2,\dots,n\}$ after the (not necessarily rank-preserving) increases of incomes of individuals $k_1,k_2,\dots,k_l$ by $\tilde{y}=(\tilde{y}_1,\tilde{y}_2,\dots,\tilde{y}_n)$. Then, by formula (2), for any permutation $\varphi: N\to N$ it holds that $G(\tilde{y})=G(\tilde{y}_\varphi)$, where $\tilde{y}_\varphi=(\tilde{y}_{\varphi(1)},\tilde{y}_{\varphi(2)},\dots,\tilde{y}_{\varphi(n)})$. To put it differently, the Gini coefficient of a population does not depend on the ordering of incomes of the population, only on the incomes themselves. In particular, if $\Phi: N\to N$ is such a permutation that $\tilde{y}_\Phi=(\tilde{y}_{\Phi(1)},\tilde{y}_{\Phi(2)},\dots,\tilde{y}_{\Phi(n)})$ and $\tilde{y}_{\Phi(1)}<\tilde{y}_{\Phi(2)}<\dots<\tilde{y}_{\Phi(n)}$, then:
(i) The increase of incomes from the vector of incomes $y=(y_1,y_2,\dots,y_n)$ to the vector of incomes $\tilde{y}_\Phi=(\tilde{y}_{\Phi(1)},\tilde{y}_{\Phi(2)},\dots,\tilde{y}_{\Phi(n)})$ is a rank-preserving increase of incomes.
(ii) If the post-increase rank $\tilde{k}$ of each individual k whose income is increased from y_k to $\tilde{y}_k$ satisfies $\tilde{k}\le\frac{n(G(y)+1)+1}{2}$, then the post-increase rank $\tilde{m}$ of each individual m whose income is increased from y_m to $\tilde{y}_{\Phi(m)}$ satisfies $\tilde{m}\le\frac{n(G(y)+1)+1}{2}$ (and this is so because $\tilde{y}_\Phi$ and $\tilde{y}$ consist of the same numbers, just ordered differently). That is, if the increase of incomes from y to $\tilde{y}$ satisfies the assumptions of Claim 2, then the increase of incomes from y to $\tilde{y}_\Phi$ also satisfies the assumptions of Claim 2.
(iii) $G(\tilde{y})=G(\tilde{y}_\Phi)$. In particular, if $G(\tilde{y}_\Phi)\le G(y)$, then $G(\tilde{y})\le G(y)$. That is, if the increase of incomes from y to $\tilde{y}_\Phi$ satisfies Claim 2, then the increase of incomes from y to $\tilde{y}$ also satisfies Claim 2.

Summing up the "to begin with" part of this proof: to prove that Claim 2 holds for the (not necessarily rank-preserving) increase of incomes from y to $\tilde{y}$, it is sufficient to prove that Claim 2 holds for the rank-preserving increase of incomes from y to $\tilde{y}_\Phi$. Thus, if Claim 2 holds for all rank-preserving increases of incomes, then it also holds for all increases of incomes.

Henceforth, without loss of generality, we assume that the increase of incomes from the vector of incomes $y=(y_1,y_2,\dots,y_n)$ to the vector of incomes $\tilde{y}=(\tilde{y}_1,\tilde{y}_2,\dots,\tilde{y}_n)$ is rank-preserving, that is, that $y_1<y_2<\dots<y_n$ and $\tilde{y}_1<\tilde{y}_2<\dots<\tilde{y}_n$. We prove the claim by induction on l, that is, on the number of the individuals who receive rank-preserving increases of their incomes.

The base case of the induction, that is, the case of $l=1$, is provided by Claim 1.

For the inductive step, as per the induction hypothesis, we assume that Claim 2 holds for $l_0\in\{1,2,\dots,n-1\}$, that is, if $k_i\le\frac{n(G(y)+1)+1}{2}$ for $i\in\{1,2,\dots,l_0\}$ and the income vector of population $N=\{1,2,\dots,n\}$ after the rank-preserving increases of the incomes of individuals $k_1,k_2,\dots,k_{l_0}$ is $\tilde{y}$, then $G(\tilde{y})\le G(y)$. For the inductive step to hold, we need to show that Claim 2 holds for $l=l_0+1$. To this end, we assume that $k_i\le\frac{n(G(y)+1)+1}{2}$ for

$i \in \{1,2,\ldots,l_0+1\}$. We denote by $\hat{y}$ the income vector of population $N=\{1,2,\ldots,n\}$ after the rank-preserving increase of the incomes of individuals $k_1,k_2,\ldots,k_{l_0},k_{l_0+1}$, so that we will then need to show that $G(\hat{y}) \le G(y)$.

Without loss of generality, we assume that $k_1<k_2<\ldots<k_{l_0}<k_{l_0+1}$ and that the income vector of population $N=\{1,2,\ldots,n\}$ after the rank-preserving increase of the incomes of individuals $k_2,k_3,\ldots,k_{l_0},k_{l_0+1}$ is $\bar{y}$ (these individuals are the individuals of the set $\{k_1,k_2,\ldots,k_{l_0},k_{l_0+1}\}$ except for individual k_1). By the induction hypothesis, $G(\bar{y}) \le G(y)$ because the number of individuals in the set $\{k_2,\ldots,k_{l_0},k_{l_0+1}\}$ is l_0. From the assumption that $k_i \le \frac{n(G(y)+1)+1}{2}$ for $i \in \{1,2,\ldots,l_0+1\}$, we also know that $k_1 \le \frac{n(G(y)+1)+1}{2}$. We consider a function f that measures the effect of the income of individual k_1 on the Gini coefficient of the population:

$$f:[y_{k_1},\hat{y}_{k_1}]: x \to G(\bar{y}_1,\ldots,\bar{y}_{k_1-1},x,\bar{y}_{k_1+1},\ldots,\bar{y}_n) \in \left(0,\frac{n-1}{n}\right],$$

where y_{k_1} denotes the initial income of individual k_1; $\hat{y}_{k_1}$ denotes the income of individual k_1 after the rank-preserving increase of the incomes of individuals $k_1,k_2,\ldots,k_{l_0},k_{l_0+1}$ (that is, the income of individual k_1 as a component of income vector $\hat{y}$); and $\bar{y}_i$ (for $i \in \{1,\ldots,n\}$) denotes the income of individual i after the rank-preserving increase of the incomes of individuals $k_2,k_3,\ldots,k_{l_0},k_{l_0+1}$, but before the increase of the income of individual k_1. In particular, the function f is continuous, and $f(y_{k_1})=G(\bar{y}) \le G(y)$. Also, $(\bar{y}_1,\ldots,\bar{y}_{k_1-1},\hat{y}_{k_1},\bar{y}_{k_1+1},\ldots,\bar{y}_n)=(\hat{y}_1,\ldots,\hat{y}_{k_1-1},\hat{y}_{k_1},\hat{y}_{k_1+1},\ldots,\hat{y}_n)=\hat{y}$. Thus, $f(\hat{y}_{k_1})=G(\hat{y})$.

To show that $G(\hat{y}) \le G(y)$, we resort to a proof by contradiction. To this end, we assume that $G(\hat{y})>G(y)$. Then

$$f(y_{k_1}) \le G(y) < G(\hat{y}) = f(\hat{y}_{k_1}),$$

and, thus, by the Darboux theorem,[7] there exists $x^* \in [y_{k_1},\hat{y}_{k_1})$ such that $f(x^*)=G(y)$. We use the notation $y^*=(\bar{y}_1,\ldots,\bar{y}_{k_1-1},x^*,\bar{y}_{k_1+1},\ldots,\bar{y}_n)$. Then $G(y^*)=G(y)$ and $k_1 \le \frac{n(G(y)+1)+1}{2} = \frac{n(G(y^*)+1)+1}{2}$. Therefore, from Claim 1 applied to income vector y^* and individual k_1 we obtain that when the income vector of the population is y^*, a rank-preserving increase of the income of individual k_1 does not increase the Gini coefficient. In particular, $x^*<\hat{y}_{k_1}$, implying that $G(\hat{y}) \le G(y^*)=G(y)$, which contradicts the assumption made at the outset of this paragraph that $G(\hat{y})>G(y)$. Therefore, $G(\hat{y}) \le G(y)$ holds, and both the inductive step and the claim itself hold.
Q.E.D.

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[7] The Darboux theorem states that if $g:[a,b] \to \mathbb{R}$ is a continuous function, $x_0 \in \mathbb{R}$, and $g(a)<x_0<g(b)$, then there exists $c \in (a,b)$ such that $g(c)=x_0$. Here, either $f(y_{k_1})=G(y)$ and then $x^*=y_{k_1}$, or $f(y_{k_1})<G(y)$, and then from the Darboux theorem we straightforwardly obtain that $x^* \in (y_{k_1},\hat{y}_{k_1})$. In sum: $x^* \in (y_{k_1},\hat{y}_{k_1}]$.